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Real algebraic morphisms represent few homotopy classes

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Abstract We study the problem of representing homotopy classes of maps between real algebraic varieties of regular maps.

1 Introduction

All real algebraic varieties that appear in the present paper are assumed to be affine (i.e., isomorphic to algebraic subsets of $\mathbb{R}^n$ for some n). Morphisms between real algebraic varieties are called regular maps. For background material on real algebraic geometry the reader may consult [6]. Every real algebraic variety carries also the Euclidean topology, induced by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

In this paper, we study the problem of representing homotopy classes of maps between real algebraic varieties by regular maps. Our results show scarcity of regular maps.

Given a real algebraic variety Y, we define a numerical invariant $\beta(Y)$ to be the supremum of all nonnegative integers n with the following property: for every n-dimensional compact connected nonsingular real algebraic variety X, every continuous map from X into Y is homotopic to a regular map.

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If d is an integer satisfying $0 \leq d \leq \beta(Y)$, then every continuous map from any d-dimensional compact connected nonsingular real algebraic variety into Y is homotopic to a regular map. Indeed, this follows from a simple fact: if A and B are real algebraic varieties and a continuous map $f : A \times B \to Y$ is homotopic to a regular map, then the restriction $f|A \times \{b\} : A \times \{b\} \to Y$ has the same property for all b in B. It is proved below that $\beta(Y) \leq \dim Y$, provided Y is compact, nonsingular, and $\dim Y \geq 1$. Of course, $\beta(Y) = \infty$ if Y is contractible. In fact our results contain a lot more information concerning possible values of $\beta(Y)$. First, we need some notation.

Assuming that Y is compact, denote by $H_k^{alg}(Y, \mathbb{Z}/2)$ the subgroup of $H_k(Y, \mathbb{Z}/2)$ of homology classes represented by k-dimensional algebraic subsets of Y, cf. [3,5,6,9]. The experts can easily verify $H_i^{alg}(Y, \mathbb{Z}/2) = H_i(Y, \mathbb{Z}/2)$ for $i \leq \beta(Y)$ (cf. also Lemma 2.10). Hence $H_k^{alg}(Y, \mathbb{Z}/2) \neq H_k(Y, \mathbb{Z}/2)$ implies

$$\beta(Y) \leq k - 1.$$

In particular, if Y is irreducible and disconnected, then one can take $k = \dim Y$ in the inequality above, and hence $\beta(Y) \leq \dim Y - 1$.

There is also a much subtler relationship between $\beta(Y)$ and the groups $H_i^{alg}(Y, \mathbb{Z}/2)$.

Theorem 1.1 *Let Y be a p-dimensional compact nonsingular real algebraic variety. If $H_{p-k}^{alg}(Y, \mathbb{Z}/2) \neq 0$ for some $k \geq 1$, then $\beta(Y) \leq k$. In particular, $\beta(Y) \leq$* $\dim Y$*, provided* $\dim Y \geq 1$.

Available also is a suitable version of Theorem 1.1 with Y noncompact. An algebraic one-point compactification of Y is a compact real algebraic variety $Y^\bullet$ containing a point $\bullet$ such that $Y^\bullet \setminus \{\bullet\}$ is biregularly isomorphic to Y (cf. [6, Proposition 3.5.3] for the existence of $Y^\bullet$).

Theorem 1.1′ *Let Y be a p-dimensional nonsingular real algebraic variety. Assume that Y is noncompact and let $Y^\bullet$ be an algebraic one-point compactification of Y. If $H_{p-k}^{alg}(Y^\bullet, \mathbb{Z}/2) \neq 0$ for some k satisfying $0 < k < p$, then $\beta(Y) \leq k$. Moreover, $\beta(Y) \leq$* $\dim Y$*, provided* $\dim Y \geq 1$ *and Y has a compact connected component.*

Certain upper bounds for $\beta(Y)$ require neither compactness nor nonsingularity of Y.

Theorem 1.2 *If on a real algebraic variety Y there is an algebraic vector bundle with nonzero kth Stiefel–Whitney class for some $k \geq 1$, then $\beta(Y) \leq k$.*

For Y nonsingular, Theorem 1.2 sometimes yields an upper bound for $\beta(Y)$ expressed in purely topological terms.

Corollary 1.3 *If Y is a nonsingular real algebraic variety whose kth Stiefel-Whitney class is nonzero for some $k \geq 1$, then $\beta(Y) \leq k$. In particular, $\beta(Y) = 0$ or $\beta(Y) = 1$, provided Y is nonorientable as a smooth manifold.*

Proof The tangent bundle to Y is algebraic, and hence the first assertion is a special case of Theorem 1.2 The second assertion follows since Y is nonorientable whenever the first Stiefel–Whitney class of Y is nonzero. □

We have other upper bounds for $\beta(Y)$ determined by the topology of Y alone.

Theorem 1.4 *If Y is a p-dimensional compact nonsingular real algebraic variety with $H_k(Y, \mathbb{Z}/2) \neq 0$ for some k satisfying $0 < k < p$, then*

$$\beta(Y) \leq \begin{cases} \max\{k, p-k\} - 1 \leq p-2 & \text{for } k \neq p/2 \\ p/2 & \text{for } k = p/2. \end{cases}$$

Our next result is of a different nature. By Tognoli's theorem [23] (cf. also [6, Theorem 14.1.10] and, for a weaker but influential result, [20]), each compact smooth (of class C^∞) manifold M is diffeomorphic to a nonsingular real algebraic variety, called an *algebraic model* of M.

Theorem 1.5 *For any compact connected smooth manifold M, given a nonnegative integer k, the following conditions are equivalent*:

(a) *k is the smallest integer such that M has an algebraic model Y with $\beta(Y) = k$,*
(b) *$\pi_i(M)$ is trivial for $1 \leq i \leq k$ and $\pi_{k+1}(M)$ is nontrivial.*

As usual, $\pi_j(M)$ stands for the jth homotopy group of M. In particular, we immediately obtain the following:

Corollary 1.6 *A compact connected smooth manifold M has an algebraic model Y with $\beta(Y) = 0$ if and only if the fundamental group of M is nontrivial.*

In general, it is hard to determine the exact value of $\beta(Y)$ even for "simple" varieties Y. Recall that a real algebraic variety of dimension n is said to be *rational* if it is irreducible and birationally equivalent to real projective n-space $\mathbb{P}^n(\mathbb{R})$. Clearly, the unit n-sphere S^n is rational for $n \geq 1$.

Example 1.7 Let Y be a nonsingular real algebraic variety.

(i) Assume Y is rational of positive dimension. Then [8, Theorem 1.1] implies $\beta(Y) \geq 1$. Hence by Corollary 1.3, $\beta(Y) = 1$, provided Y is nonorientable. Theorem 1.1 implies $\beta(\mathbb{P}^n(\mathbb{R})) = 1$ for all $n \geq 1$ since $H_i^{alg}(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) = H_i(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2)$ for $i \geq 0$.
(ii) Assume Y is homeomorphic to S^n, where $n \geq 2$. Then $n - 1 \leq \beta(Y) \leq n$, the lower bound being an obvious topological fact. No example with $\beta(Y) = n$ is known. If n is even, then $\beta(S^n) = n - 1$, the possibility $\beta(S^n) = n$ being excluded by [7, Theorem 2.4].
(iii) If $0 < k < l$ and Y is homeomorphic to $S^k \times S^l$, then $k - 1 \leq \beta(Y) \leq l - 1$, where the lower bound is obvious, while the upper bound follows from Theorem 1.4. For the same reason, if Y is homeomorphic to $S^k \times S^k$, then $k - 1 \leq \beta(Y) \leq k$.

(iv) Omitting the nonsingularity assumption, suppose Y is a p-dimensional algebraic subset of $\mathbb{P}^n(\mathbb{R})$. If $p \geq 1$ and Y represents a nonzero homology class in $H_p(\mathbb{P}^n(R), \mathbb{Z}/2)$, then $\beta(Y) = 0$ or $\beta(Y) = 1$. Indeed, one readily sees that $i^*(v) \neq 0$, where $i : Y \hookrightarrow \mathbb{P}^n(\mathbb{R})$ is the inclusion map and v is the unique generator of $H^1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2) \cong \mathbb{Z}/2$. Moreover, if γ_n is the universal line bundle on $\mathbb{P}^n(\mathbb{R})$, then the first Stiefel–Whitney class of $\gamma_n|X$ is equal to $i^*(v)$. Hence, Theorem 1.2 implies $\beta(Y) \leq 1$, as asserted.

(v) In view of (i), $\beta(S^1) = 1$. We conjecture that, up to isomorphism, the only variety Y (of arbitrary dimension) with $\beta(Y) = \dim Y$ is S^1. If C is a compact irreducible nonsingular real algebraic curve which is not biregularly isomorphic to S^1, then $\beta(C) = 0$ (each regular map from S^1 into C is constant). More generally, $\beta((S^1)^n) = 1$ and $\beta(C^n) = 0$ for the n-fold products of S^1 and C, respectively.

The case of surfaces deserves to be singled out.

Theorem 1.8 *For any compact nonsingular real algebraic surface Y, one has $\beta(Y) = 0$ or $\beta(Y) = 1$. If Y is either rational or homeomorphic to S^2, then $\beta(Y) = 1$.*

It is not known if $\beta(Y) = 1$ is possible for compact surfaces Y other than these listed in the second part of Theorem 1.8. On the other hand, we have the following:

Corollary 1.9 *A compact connected smooth manifold M of dimension 2 is homeomorphic to S^2 if and only if $\beta(Y) = 1$ for all algebraic models Y of M.*

Proof It is sufficient to apply Corollary 1.6 and Theorem 1.8. □

Going beyond surfaces, the picture is less complete.

Example 1.10 Let Y be a compact nonsingular real algebraic variety.

(i) If $\dim Y = 3$ and $H_1(Y, \mathbb{Z}/2) \neq 0$, then by Theorem 1.4, $\beta(Y) = 0$ or $\beta(Y) = 1$. If in addition Y is rational, then $\beta(Y) = 1$ in view of Example 1.7 (i).

(ii) Assume $\dim Y = 4$, Y is connected and simply connected. Then $\beta(Y) \geq 3$ if and only if Y is homeomorphic to S^4. Indeed, suppose $\beta(Y) \geq 3$. By Theorem 1.4, $H_2(Y, \mathbb{Z}/2) = 0$, and hence the group $H_2(Y, \mathbb{Z})$ is of rank 0. According to [13, Theorem 1.6], Y is homeomorphic to S^4. The converse is obvious.

All theorems announced above are proved in Sect. 2, which also contains other results. Among the latter Theorem 2.8 is noteworthy, being a much sharper (if somewhat technical) version of Theorems 1.1 and 1.1′ combined. Of independent interest is Theorem 2.7, obtained as a free byproduct and not directly connected to the problem under consideration.

2 Proofs and other results

To begin with, we consider some topological preliminaries. Given a compact smooth manifold N of dimension n, we denote by $[N]$ its fundamental class in $H_n(N, \mathbb{Z}/2)$. We say that N is a *boundary* if it is diffeomorphic to the boundary of a compact smooth manifold with boundary. Boundaries will play an important role, and in particular the following the version of Thom's representability theorem [22] will be essential.

Theorem 2.1 *Let Y be a CW-complex and let β be a homology class in $H_k(Y, \mathbb{Z}/2)$ with $k \geq 1$. Then, there exist a k-dimensional compact smooth manifold K and a continuous map $h : K \to Y$ such that $h_*([K]) = \beta$ and K is a boundary. Moreover, K can be chosen connected, provided Y is connected.*

Proof Without loss of generality, we may assume that Y is compact and connected. Hence, we can find a compact connected smooth manifold P, with $\dim P = p \geq 2k + 1$, containing Y as a retract, cf [22, pp. 57,58]. Let $i : Y \hookrightarrow P$ be inclusion map and let $r : P \to Y$ be a retraction. By Thom's theorem [22, Théorème II. 26], the homology class $i_*(\beta)$ in $H_k(P, \mathbb{Z}/2)$ is represented by a k-dimensional compact smooth submanifold N of P. Since P is connected and $k \geq 1$, joining the connected components of N with k-dimensional tubes, we may assume that N is connected. Let U be an open subset of $P \setminus N$ diffeomorphic to $\mathbb{R}^p$. There is a smooth submanifold N' of U diffeomorphic to N, the integer p being sufficiently large. By joining N and N' with a k-dimensional tube, we obtain a compact connected smooth submanifold K of P that represents $i_*(\beta)$. Moreover, K is a boundary, being diffeomorphic to the connected sum $N \# N$. Define $h : K \to Y$ to be the composition $r \circ j$, where $j : K \hookrightarrow P$ is the inclusion map. Since $j_*([K]) = i_*(\beta)$ and $r \circ i$ is the identity map of Y, we get

$$h_*([K]) = (r \circ j)_*([K]) = r_*(j_*([K])) = r_*(i_*(\beta)) = (r \circ i)_*(\beta) = \beta,$$

which completes the proof. □

Theorem 2.1 is applicable if Y is a real algebraic variety, since all such varieties are triangulable, cf. [6, Theorem 9.2.1]. For the intended applications, we need certain constructions from real algebraic geometry.

Given a compact nonsingular real algebraic variety B, we set

$$H^k_{alg}(B, \mathbb{Z}/2) := D_B^{-1}(H_l^{alg}(B, \mathbb{Z}/2)),$$

where $k + l = \dim B$ and $D_B : H^k(B, \mathbb{Z}/2) \to H_l(B, \mathbb{Z}/2)$ is the Poincaré duality isomorphism. The expected functorial property holds: If $h : A \to B$ is a regular map between compact nonsingular real algebraic varieties, then

$$h^*(H^k_{alg}(B, \mathbb{Z}/2)) \subseteq H^k_{alg}(A, \mathbb{Z}/2), \tag{$*$}$$

where $h^* : H^k(B, \mathbb{Z}/2) \to H^k(A, \mathbb{Z}/2)$ is the induced homomorphism, cf. [3,5,9].

The definition of $H^k_{alg}(-,\mathbb{Z}/2)$ can be extended to arbitrary real algebraic varieties. For any real algebraic variety Y, a cohomology class v in $H^k(Y,\mathbb{Z}/2)$ is said to be *algebraic* if there exist a compact nonsingular real algebraic variety B, a regular map $\lambda : Y \to B$, and a cohomology class b in $H^k_{alg}(B,\mathbb{Z}/2)$ such that $v = \lambda^*(b)$. We write $H^k_{alg}(Y,\mathbb{Z}/2)$ for the set of all algebraic cohomology classes in $H^k(Y,\mathbb{Z}/2)$. In view of $(*)$, this notation is consistent with the one previously introduced for compact nonsingular real algebraic varieties.

Proposition 2.2 *For any real algebraic variety Y, the set $H^k_{alg}(Y,\mathbb{Z}/2)$ is a subgroup of $H^k(Y,\mathbb{Z}/2)$. If $f : X \to Y$ is a regular map between real algebraic varieties, then $f^*(H^k_{alg}(Y,\mathbb{Z}/2)) \subseteq H^k_{alg}(X,\mathbb{Z}/2)$.*

Proof In order to prove the first assertion, it is sufficient to show that the set $H^k_{alg}(Y,\mathbb{Z}/2)$ is closed under addition. Suppose v_i is in $H^k_{alg}(Y,\mathbb{Z}/2)$ for $i = 1,2$. There exist a compact nonsingular real algebraic variety B_i, a regular map $\lambda_i : Y \to B_i$, and a cohomology class b_i in $H^k_{alg}(B_i,\mathbb{Z}/2)$ such that $v_i = \lambda_i^*(b_i)$. Define $\lambda : Y \to B_1 \times B_2$ by $\lambda(x) = (\lambda_1(x), \lambda_2(x))$ for all x in Y. Denoting by $\pi_i : B_1 \times B_2 \to B_i$ the canonical projection, we get $\lambda_i = \pi_i \circ \lambda$, and hence

$$\begin{aligned} v_1 + v_2 &= \lambda_1^*(b_1) + \lambda_2^*(b_2) = (\pi_1 \circ \lambda)^*(b_1) + (\pi_2 \circ \lambda)^*(b_2) \\ &= \lambda^*(\pi_1^*(b_1) + \pi_2^*(b_2)). \end{aligned}$$

In view of $(*)$, $\pi_1^*(b_1) + \pi_2^*(b_2)$ is in $H^k_{alg}(B_1 \times B_2,\mathbb{Z}/2)$, which shows that $v_1 + v_2$ is in $H^k_{alg}(Y,\mathbb{Z}/2)$.

The second assertion follows directly from the definition of algebraic cohomology classes. □

For Y nonsingular and noncompact, the group $H^k_{alg}(Y,\mathbb{Z}/2)$ can be computed as follows.

Proposition 2.3 *Let Y be a p-dimensional nonsingular real algebraic variety. Assume that Y is noncompact and let $Y^\bullet$ be an algebraic one-point compactification of Y. For each k, with $0 \le k \le p-1$, there is an isomorphism $\varphi_k : H^k(Y,\mathbb{Z}/2) \to H_{p-k}(Y^\bullet,\mathbb{Z}/2)$ transforming $H^k_{alg}(Y,\mathbb{Z}/2)$ onto $H^{alg}_{p-k}(Y^\bullet, \mathbb{Z}/2)$, while $H^p_{alg}(Y,\mathbb{Z}/2) = H^p(Y,\mathbb{Z}/2)$.*

Proof Given a real algebraic variety V, denote by $\overline{H}_l(V,\mathbb{Z}/2)$ the lth Borel-Moore homology group of V and by $\overline{H}_l^{alg}(V,\mathbb{Z}/2)$ its subgroup consisting of the homology classes represented by l-dimensional algebraic subsets of V. If V is compact, then $\overline{H}_l(V,\mathbb{Z}/2) = H_l(V,\mathbb{Z}/2)$ and $\overline{H}_l^{alg}(V,\mathbb{Z}/2) = H_l^{alg}(V,\mathbb{Z}/2)$. Suppose that V is noncompact and let $V^\bullet$ be an algebraic one-point compactification of V. Since $V^\bullet$ is triangulable, the groups $\overline{H}_l(V,\mathbb{Z}/2)$ and $H_l(V^\bullet,\mathbb{Z}/2)$ can be identified for $l \ge 1$; under this identification $\overline{H}_l^{alg}(V,\mathbb{Z}/2)$ corresponds to $H_l^{alg}(V^\bullet,\mathbb{Z}/2)$. Clearly, $\overline{H}_0^l(V,\mathbb{Z}/2) = \overline{H}_0^{alg}(V,\mathbb{Z}/2)$.

For all these facts the reader may consult [9].

Assuming V is nonsingular and $\dim V = p$, set

$$\overline{H}^k_{alg}(V,\mathbb{Z}/2) = D_V^{-1}(\overline{H}^{alg}_{p-k}(V,\mathbb{Z}/2)),$$

where $D_V : H^k(V,\mathbb{Z}/2) \to \overline{H}_{p-k}(V,\mathbb{Z}/2)$ is the Poincaré duality isomorphism. In view of the remarks above, it suffices to show

(i) $$\overline{H}^k_{alg}(V,\mathbb{Z}/2) = H^k_{alg}(V,\mathbb{Z}/2)$$

for $k \geq 0$. This can be done as follows. According to [9], $\overline{H}^k_{alg}(-,\mathbb{Z}/2)$ is functorial: If $f : V \to W$ is a regular map between nonsingular real algebraic varieties, then

(ii) $$f^*(\overline{H}^k_{alg}(W,\mathbb{Z}/2)) \subseteq \overline{H}^k_{alg}(V,\mathbb{Z}/2).$$

Clearly, (i) holds if V is compact, and hence (ii) implies $H^k_{alg}(V,\mathbb{Z}/2) \subseteq \overline{H}^k_{alg}(V,\mathbb{Z}/2)$ for any nonsingular V.

Now it remains to justify $\overline{H}^k_{alg}(V,\mathbb{Z}/2) \subseteq H^k_{alg}(V,\mathbb{Z}/2)$. By Hironaka's resolution of singularities theorem [15], we may assume that V is a Zariski open subset of a compact nonsingular real algebraic variety W. We have the following commutative diagram:

$$\begin{array}{ccc} H^k(W,\mathbb{Z}/2) & \xrightarrow{i^*} & H^k(V,\mathbb{Z}/2) \\ {\scriptstyle D_W}\big\downarrow & & {\scriptstyle D_V}\big\downarrow \\ \overline{H}_{p-k}(W,\mathbb{Z}/2) & \xrightarrow{r} & \overline{H}_{p-k}(V,\mathbb{Z}/2), \end{array}$$

where $i : V \hookrightarrow W$ is the inclusion map and r is the restriction homomorphism. Since $r(\overline{H}^{alg}_{p-k}(W,\mathbb{Z}/2)) = \overline{H}^{alg}_{p-k}(V,\mathbb{Z}/2)$, we get $i^*(\overline{H}^k_{alg}(W,\mathbb{Z}/2)) = \overline{H}^k_{alg}(V,\mathbb{Z}/2)$. Moreover, $\overline{H}^k_{alg}(W,\mathbb{Z}/2) = H^k_{alg}(W,\mathbb{Z}/2)$ due to the compactness of W, and hence the required inclusion follows.

In conclusion, (i) is established and the proof is complete. □

It is often rather hard to decide whether or not the group $H^k_{alg}(Y,\mathbb{Z}/2)$ is nontrivial, which is an important question for our purposes (cf. Theorems 2.8 and 2.9). In some cases, it is helpful to consider algebraic vector bundles, whose basic properties can be found in [6].

Proposition 2.4 *The kth Stiefel–Whitney class of any algebraic vector bundle on a real algebraic variety Y is in $H^k_{alg}(Y,\mathbb{Z}/2)$ for all $k \geq 0$.*

Proof Let ξ be an algebraic vector bundle on Y. Without loss of generality, we may assume that ξ is of constant rank, say, r. Denote by $\mathbb{G}_{n,r}$ the Grassmann variety of r-dimensional vector subspaces of $\mathbb{R}^n$. It is well known that $\mathbb{G}_{n,r}$ is a compact nonsingular real algebraic variety with $H^i_{alg}(\mathbb{G}_{n,r},\mathbb{Z}/2) = H^i(\mathbb{G}_{n,r},\mathbb{Z}/2)$

for all $i \geq 0$, cf. [6, Proposition 11.3.3]. If n is large enough, there is a regular map $g : Y \to \mathbb{G}_{n,r}$ such that ξ is isomorphic to the pullback $g^*\gamma_{n,r}$ of the universal vector bundle $\gamma_{n,r}$ on $\mathbb{G}_{n,r}$, cf. [6, Theorem 12.1.7]. The proof is complete since $w_k(\xi) = g^*(w_k(\gamma_{n,r}))$, where $w_k(-)$ stands for the kth Stiefel-Whitney class. □

Let X be a compact nonsingular real algebraic variety. Define $Alg^l(X)$ to be the subset of $H^l(X, \mathbb{Z}/2)$ consisting of all elements v for which there exist a compact irreducible nonsingular real algebraic variety T (depending on v), two points t_0 and t_1 in T, and a cohomology class z in $H^l_{alg}(X \times T, \mathbb{Z}/2)$ such that

$$v = i^*_{t_1}(z) - i^*_{t_0}(z),$$

where given t in T, we let $i_t : X \to X \times T$ denote the map defined by $i_t(x) = (x, t)$ for all x in X. For properties and alternative definition of $Alg^l(X)$, the reader may refer to [18,19]. In particular, $Alg^l(X)$, is a subgroup of $H^l_{alg}(X, \mathbb{Z}/2)$, and hence

$$Alg_k(X) := D_X(Alg^l(X)),$$

where $k + l = \dim X$, is a subgroup of $H^{alg}_k(X, \mathbb{Z}/2)$. We will make use of the following result.

Proposition 2.5 *Let X be a compact nonsingular real algebraic variety. For each $k \geq 0$, if u is in $H^k_{alg}(X, \mathbb{Z}/2)$ and α is in $Alg_k(X)$, then $< u, \alpha >= 0$.*

Proof Set $v := D_X^{-1}(\alpha)$. Then $\alpha = v \cap [X]$ and

$$< u, \alpha >=< u, v \cap [X] >=< u \cup v, [X] >= 0,$$

where the last equality is proved in [18, Theorem 2.1] (cf. also [19, Theorem 4.4]). □

Of course, $< \ , \ >, \cap, \cup$ appearing above stand for the familiar products in algebraic topology, whose properties used here are all proved in [10].

Given a compact smooth manifold M and an n-dimensional smooth submanifold N of M (submanifolds are always assumed to be closed subsets), we let $[N]_M$ denote the homology class in $H_n(M, \mathbb{Z}/2)$ represented by N. As usual, $w_i(M)$ will stand for the ith Stiefel–Whitney class of M. Now we will focus on constructing algebraic models of M satisfying some additional desirable properties.

Theorem 2.6 *Let M be a compact smooth manifold. Let K be a k-dimensional smooth submanifold of M such that $0 < k < \dim M$, the normal vector bundle of K is trivial, and K is a boundary. Then there exist an irreducible algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ for which $[\varphi^{-1}(K)]_X$ is in $Alg_k(X)$.*

Proof Let $e : K \hookrightarrow M$ be the inclusion map. Setting $v := D_M^{-1}([K]_M)$, we have

$$v \cap [M] = [K]_M = e_*([K]).$$

We assert that

(i) $$< w_{i_1}(M) \cup \ldots \cup w_{i_r}(M) \cup v, [M] >= 0$$
for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = k$. The proof requires a simple computation. Setting $w = w_{i_1}(M) \cup \ldots \cup w_{i_r}(M)$, we get

$$\begin{aligned} < w \cup v, [M] > &= < w, v \cap [M] > \\ &= < w, e_*([K]) > \\ &= < e^*(w), [K] > \\ &= < e^*(w_{i_1}(M)) \cup \ldots \cup e^*(w_{i_r}(M)), [K] > \\ &= < w_{i_1}(K) \cup \ldots \cup w_{i_r}(K), [K] >, \end{aligned}$$

where the last equality holds since the triviality of the normal vector bundle of K implies $e^*(w_i(M)) = w_i(K)$ for all $i \geq 0$. Since K is a boundary, Pontryagin's theorem [22, Théorème IV.3] yields.

$$< w_{i_1}(K) \cup \ldots \cup w_{i_r}(K), [K] >= 0$$

for all nonnegative integers $i_1, \ldots, i_r$ with $i_1 + \cdots + i_r = k$. Hence (i) is established.
Let $l = \dim M - k$. Making use once again of the triviality of the normal vector bundle of K, we can represent v as

(ii) $$v = f^*(c),$$
where $f : M \to S^l$ is a smooth map and c is the unique generator of $H^l(S^l, \mathbb{Z}/2) \cong \mathbb{Z}/2$, cf. [22, Théorème II.2].

It follows from (i) and (ii) that [1, Theorem 1.2] is applicable, and therefore there exist an irreducible algebraic model X of M and a smooth diffeomorphism $\varphi : X \to M$ such that $\varphi^*(v)$ is in $Alg^l(X)$, or equivalently, $D_X(\varphi^*(v))$ is in $Alg_k(X)$. The proof is complete since $D_X(\varphi^*(v)) = [\varphi^{-1}(K)]_X$. □

As a byproduct of Theorem 2.6, not needed for the remainder of this paper but interesting in its own right, we obtain the following result.

Theorem 2.7 *Let M be a compact smooth manifold. Assume that there is a k-dimensional smooth submanifold K of M such that $0 < k < \dim M$, the normal vector bundle of K is trivial, $[K]_M \neq 0$, and K is a boundary. Then there exists an irreducible algebraic model X of M with $H^k_{alg}(X, \mathbb{Z}/2) \neq H^k(X, \mathbb{Z}/2)$.*

Proof Since $[K]_M \neq 0$, by Theorem 2.6, there is an irreducible algebraic model X of M together with a nonzero homology class α in $Alg_k(X)$. Choose a cohomology class u in $H^k(X, \mathbb{Z}/2)$ for which $< u, \alpha > \neq 0$. Proposition 2.5 implies that u does not belong to $H^k_{alg}(X, \mathbb{Z}/2)$. □

Now we are ready for the main result of this section.

Theorem 2.8 *Let K and L be compact smooth manifolds with* $\dim K = k \geq 1$ *and* $\dim L \geq 1$. *Let $\pi : K \times L \to K$ be the canonical projection. If K is a boundary, then there exist an irreducible algebraic model X of $K \times L$ and a smooth diffeomorphism $\varphi : X \to K \times L$ having the following property: for any real algebraic variety Y and any continuous map $h : K \to Y$ such that*

$$< w, h_*([K]) > \neq 0$$

for some w in $H^k_{alg}(Y, \mathbb{Z}/2)$, the continuous map $h \circ \pi \circ \varphi : X \to Y$ is not homotopic to any regular map.

Proof Let p_0 be a point in L. Since the normal vector bundle of $K \times \{p_0\}$ in $K \times L$ is trivial, by Theorem 2.6, there exist an irreducible algebraic model X of $K \times L$ and a smooth diffeomorphism $\varphi : X \to K \times L$ such that the homology class $\alpha := [\varphi^{-1}(K \times \{p_0\})]_X$ is in $Alg_k(X)$.

It remains to show that X and φ have the required property. Let Y be a real algebraic variety and let $h : K \to Y$ be a continuous map. Suppose that the map $h \circ \pi \circ \varphi : X \to Y$ is homotopic to a regular map $f : X \to Y$. We will demonstrate that h cannot satisfy the hypothesis of the theorem. Indeed,

$$h_*([K]) = h_*(\pi_*([K \times \{p_0\}])) = h_*(\pi_*(\varphi_*(\alpha))) = f_*(\alpha),$$

and hence for every cohomology class w in $H^k_{alg}(Y, \mathbb{Z}/2)$,

$$< w, h_*([K]) > = < w, f_*(\alpha) > = < f^*(w), \alpha > = 0,$$

where the last equality follows from Propositions 2.2 and 2.5. The proof is complete. □

Theorem 2.8 provides a method for constructing continuous maps which are not homotopic to any regular map. This method is quite efficient since, as demonstrated by Theorem 2.1, the assumption that K be a boundary is not at all restrictive in applications. In particular, Theorem 2.8 is significantly sharper than Theorems 1.1, 1.1′, 1.2, 1.2′ and 1.3 of [14] combined, whereas its proof is much simpler.

Theorem 2.9 *If Y is a real algebraic variety with $H^k_{alg}(Y, \mathbb{Z}/2) \neq 0$ for some $k \geq 1$, then $\beta(Y) \leq k$.*

Proof Let w be a nonzero cohomology class in $H^k_{alg}(Y, \mathbb{Z}/2)$. We can find a homology class β in $H_k(Y, \mathbb{Z}/2)$ with $< w, \beta > \neq 0$ and such that β is represented by a cycle contained in one connected component of Y. By Theorem 2.1, β can be written as $\beta = h_*([K])$, where K is a k-dimensional compact connected smooth manifold that is a boundary and $h : K \to Y$ is a continuous map. The proof is completed by applying Theorem 2.8 with $L = S^1$. □

Proof of Theorem 1.1. Since Y is compact and nonsingular, and the group $H^{alg}_{p-k}(Y,\mathbb{Z}/2)$ is nontrivial, the group $H^k_{alg}(Y,\mathbb{Z}/2)$ is nontrivial too. Hence Theorem 2.9 implies $\beta(Y) \leq k$. Clearly, $H^p_{alg}(Y,\mathbb{Z}/2) = H^p(Y,\mathbb{Z}/2) \neq 0$, which yields $\beta(Y) \leq \dim Y$, provided $\dim Y \geq 1$. □

Proof of Theorem 1.1′. It is sufficient to apply Proposition 2.3 and Theorem 2.9. □

Proof of Theorem 1.2. It suffices to apply Proposition 2.4 and Theorem 2.9. □

For sake of completeness we will prove the following, obvious to the experts, fact.

Lemma 2.10 *For any compact real algebraic variety Y, one has*

$$H_i^{alg}(Y,\mathbb{Z}/2) = H_i(Y,\mathbb{Z}/2)$$

for $i \leq \beta(Y)$.

Proof By Thom's representability theorem [22], the group $H_i(Y,\mathbb{Z}/2)$ is generated by homology classes of the form $h_*([A])$, where A is an i-dimensional compact connected smooth manifold and $h : A \to Y$ is a continuous map. Since A has an algebraic model, we can assume that A itself is a nonsingular real algebraic variety. If $i \leq \beta(Y)$, then h is homotopic to a regular map, say, $f : A \to Y$. Hence the homology class $h_*([A]) = f_*([A])$ is in $H_i^{alg}(Y,\mathbb{Z}/2)$, cf. [3,5,9]. This proves $H_i^{alg}(Y,\mathbb{Z}/2) = H_i(Y,\mathbb{Z}/2)$, as required. □

Proof of Theorem 1.4. Assume $H_k(Y,\mathbb{Z}/2) \neq 0$ for some k satisfying $0 < k < p$. By duality, $H_{p-k}(Y,\mathbb{Z}/2) \neq 0$. Set $l = \max\{k, p-k\}$ and suppose $\beta(Y) \geq l$. Lemma 2.10 implies $H_i^{alg}(Y,\mathbb{Z}/2) = H_i(Y,\mathbb{Z}/2)$ for $0 \leq i \leq l$. Since $k \geq 1$ and $p-k \geq 1$, Theorem 1.1 yields $\beta(Y) \leq \min\{k, p-k\}$. If $k \neq p/2$, we get a contradiction, and hence $\beta(Y) \leq l-1$, as required. If $k = p/2$, then $\beta(Y) \leq p/2$. The proof is complete. □

Proof of Theorem 1.8. First suppose Y is rational. Up to biregular isomorphism, Y is obtained from S^2 or $S^1 \times S^1$ by successively blowing up finitely many (possibly zero) points (cf. [11,12] and for a modern treatment [17, proof of Theorem 2.3] or [21, proof of Proposition 6.4, p. 137]). In particular, Y is orientable if and only if it is biregularly isomorphic to either S^2 or $S^1 \times S^1$. Hence $\beta(Y) = 1$ in view of Example 1.7 (i), (ii), (v).

Now assume that Y is homeomorphic to S^2. We already know that $1 \leq \beta(Y) \leq 2$. Suppose $\beta(Y) = 2$. This implies the existence of a regular map $f : S^2 \to Y$ of topological degree 1. By Hironaka's resolution of singularities theorem [15], we may assume that Y is the set $V(\mathbb{R})$ of real points of an irreducible nonsingular complex projective surface V defined over $\mathbb{R}$.

Since f is surjective, it follows that V is unirational, and hence rational over $\mathbb{C}$, cf. [16, p.170, Theorem 2.4]. Actually, V is rational over $\mathbb{R}$, the set $V(\mathbb{R}) = Y$ being connected, cf. [21, p.137, Corollary 6.5]. Therefore, Y is a rational real algebraic surface. We already proved $\beta(Y) = 1$ in such a case, which contradicts the hypothesis $\beta(Y) = 2$.

Next we do not impose any restriction on Y, except irreducibility, which can be done without loss of generality. If Y is disconnected, then $\beta(Y) \leq 1$, according to the observation preceding Theorem 1.1. Suppose Y is connected. If Y is not homeomorphic to S^2, then $H_1(Y, \mathbb{Z}/2) \neq 0$, and hence Theorem 1.4 implies $\beta(Y) \leq 1$. The case of Y homeomorphic to S^2 is already settled. □

Theorem 2.11 *Any compact connected smooth manifold M has an algebraic model Y such that every regular map from any rational real algebraic variety into Y is constant.*

Proof Let C be a compact connected nonsingular real algebraic curve that is not biregularly isomorphic to S^1. Then, every regular map from any rational real algebraic variety into C is constant.

Fix an integer n satisfying $n \geq 2 \dim M + 1$ and let $e : M \to C^n$ be a smooth embedding. Endow the space $C^\infty(M, C^n)$ of smooth maps from M into C^n with the C^∞ topology. Let $\mathcal{V}$ be a neighborhood of e in $C^\infty(M, C^n)$ consisting of smooth embeddings. Since C is connected, the homology group $H_*(C^n, \mathbb{Z}/2)$ is generated by the homology classes of nonsingular algebraic subsets of C^n. Hence by [2, Proposition 2.8] or [4, Theorem 3], there exist an algebraic model Y of M, a smooth diffeomorphism $\psi : M \to Y$, and a regular map $r : Y \to C^n$ such that $r \circ \psi$ is in $\mathcal{V}$. In particular, $r : Y \to C^n$ is a smooth embedding, and hence injective.

We claim that Y satisfies the required condition. Let X be a rational real algebraic variety and let $f : X \to Y$ be a regular map. The map $r \circ f : X \to C^n$ is regular, and therefore constant. It follows that f is constant, r being injective. □

Proof of Theorem 1.5 First recall a standard fact from topology. Let P be a connected manifold and let r be a positive integer. The homotopy groups $\pi_i(P)$ are trivial for $1 \leq i \leq r$ if and only if for every compact polyhedron X of dimension at most r, every continuous map from X into P is homotopic to a constant map, cf. [10, p.459, Corollary 7.3, p.509, Corollary 13.14].

Suppose (b) is satisfied. Let Y be an algebraic model of M as in Theorem 2.11. It follows from the remark above that $\beta(Y) = k$, while $\beta(Z) \geq k$ for all algebraic models Z of M. Hence (a) holds.

The same arguments also shows that (a) implies (b). □

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